The Cantor set.

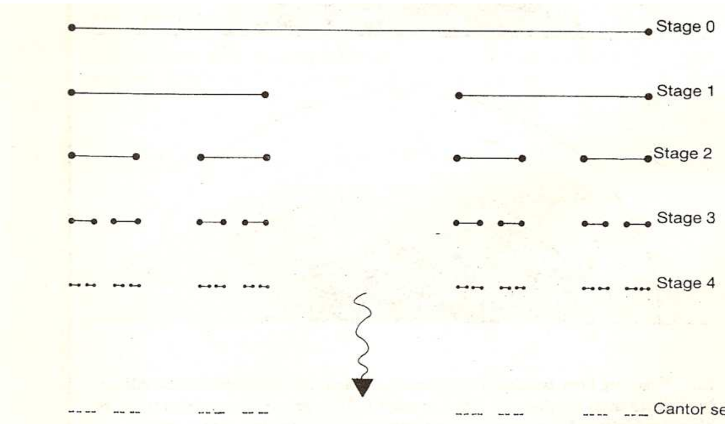
(This is a conversation between Henry and Anne-Lida.)

- *Henry* : Georg Cantor was a German mathematician who invented a very curious set in about 1883 […]To get a Cantor set you start with a line segment of length 1, and remove its middle third. Now remove the middle third of each remaining piece. Repeat, forever. What is left is the Cantor set. (Figure 1.6)
- *Anne-Lida* : I don't see how there can be *anything* left, Henry.
- *Henry* :Oh, but there is. All the end-points of all the smaller segments are left, for a start. And many others. But you are right in one way, my dear. What is the length of the Cantor set ?
- *Anne-Lida* :Its ends are distance 1 apart, Henry.
- *Henry* :No, I meant the length not counting the gaps.
- *Anne-Lida* :I have no idea, Henry. But it looks very small to me. The set is mostly holes.
- […]
- *Henry* :The length reduces to 2/3 the size at each stage, so the total length after the *n*th stage is $\left(\frac{2}{3}\right)^n$. As *n* tends to infinity, this tends to 0. The length of the Cantor set is zero.

Ian Stewart, « Game, Set & Math, enigmas and conundrums », Dover publications, 2007.

Stage	Segment length	Number of segments	Total set length	Number of end-points
0	1	1	1	2
1	$\frac{1}{3}$	2	$2\times\frac{1}{3}=\frac{2}{3}$	2^2
2				
3				
4				

Table 1



1.6 Construction of the Cantor set by removing middle thirds. Its length is zero, but contains infinitely many points.

TASKS :

1) Fill in table 1 above with integers or fractions raised to the appropriate power.
2) Conjecture the line of table 1 at stage n.
3) Explain why the Cantor set at stage n contains more than $2^{(n+1)}$ points.
4) Explain why the Cantor set at stage n has length $\left(\frac{2}{3}\right)^n$.
5) Let n go to infinity and give the length of the Cantor set. Justify your answer.
6) How many points are contained in this Cantor set ?
7) Using the preceeding results, do you think it is possible for a set to be simultaneously of length zero and non empty ? Explain your reasoning.

The Cantor set.
ANSWERS

1)

Stage	Segment length	Number of segments	Number of end-points	Total set length
0	1	1	2	1
1	1/3	2	2^2	2*1/3=2/3
2	1/3*1/3=(1/ 3)^2	2*2=2^2	2^3	2^2*(1/3)^2=(2/3)^2
3	1/3*(1/3)^2=(1:3)^3	2^3	2^4	2^3*(1/3)^3=(2/3)^3
4	1/3*(1/3)^3=(1/3)^4	2^4	2^5	(2/3)^4

…..
2)
n (1/3)^n 2^n 2^(n+1) (2/3)^n

3) the number of end-points is doubled at each stage starting with 2 points at stage 0. Therefore, this number of points is a geometric sequence with constant 2 and 2 as first term. At stage n, the number of end-points contained in the set is 2^(n+1) and the set contains the points of the small line segments determined by these end-points too.

4) The length of the Cantor set is
(the number of line segments) x (the segment length)
= 2^n x (1/3)^n
= (2/3)^n

5) As n goes to infinity, the Cantor set has length zero, using the limit of a geometric sequence whose constant q satisfies 0<q<1

6) As n goes to infinity, 2^(n+1) goes to infinity, since 2>1. Therefore, the Cantor set contains infinetely many points.

7) Thus, the Cantor set shows that it is possible for a set to be of length 0 and non empty, which is contradicting our common sense. This set contains infinitely many points since 2^(n+1) goes to infinity when n goes to infinity.